Calculus II - Chapter 2.6 - Alternating Series; Absolute Convergence, and Conditional Convergence: (Can apply to positive and negative-term series)

(What if series have negative and postive terms? No worries... It's easy!)

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1. **Definition: (Alternating Series)** An **alternating series** is an infinite series of the form

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} c_n = c_1 - c_2 + c_3 - c_4 + c_5 - \cdots$$

or

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n} c_n = -c_1 + c_2 - c_3 + c_4 - c_5 + \cdots$$

where $c_n > 0$ for all n. So,

$$c_n = |a_n| = \textbf{non-alternating part of the series.}$$

2. **Theorem: Alternating Series Test**

Given an alternating series of the form $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} c_n$ or $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n} c_n$, let $c_n > 0$ for all n. If

1. $c_n \geq c_{n+1} > 0$ for all n (ie. the sequence $\{c_n\}$ is eventually decreasing for all n from some value n), and

2. $\lim_{n\to\infty} c_n = 0$.

Then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} c_n$ or $\sum_{n=1}^{\infty} (-1)^{n} c_n$ converge.

Note 1: The *n-th term test* applies to all series: If the terms of any series do not tend to zero, then the series diverges. Hence, if $\lim_{n\to\infty} c_n \neq 0$, the alternating series diverges. Also, if the sequence $\{c_n\}$ is not decreasing, then the alternating series diverges.

Note 2. (The common confusion): For series with positive terms only, $\lim_{n\to\infty} a_n = 0$ **does not** imply convergence (n-th term test). However, for the alternating series, $\lim_{n\to\infty} c_n = 0$ **does** imply convergence.

Proof of the theorem:

Consider the alternating series of the form $\sum_{n=1}^{\infty}(-1)^{n+1}c_n = c_1 - c_2 + c_3 - c_4 + c_5 - \cdots$.

By assumption, the terms of the series decrease in size. Denote

$\{s_{2n}\}$=the even terms of the sequence of partial sums (s_2, s_4, $s_6, ...$), and
$\{s_{2n-1}\}$=the odd terms of the sequence of partial sums (s_1, s_3, $s_5, ...$).

Note that since the sequence is decreasing, $\{s_{2n}\}$ is bounded above by s_1. Hence, by the *Bounded Monotonic Sequence Theorem* (chapter 2.1), this sequence must have a limit; call it L. Similarly, the odd terms of the sequence of partial sums $\{s_{2n-1}\}$form a non-increasing sequence that is bounded below by s_2. By the *Bounded Monotonic Sequence Theorem* (chapter 2.1), this sequence must have a limit; call it L'.

Recall that since the series is alternating, we have
$s_1 = c_1$
$s_2 = c_1 - c_2 = s_1 - c_2$
$s_3 = c_1 - c_2 + c_3 = s_2 + c_3$
$s_4 = c_1 - c_2 + c_3 - c_4 = s_3 - c_4$ and so forth. Hence,

$s_2 - s_1 = (c_1 - c_2) - c_1 = -c_2$
$s_4 - s_3 = (c_1 - c_2 + c_3 - c_4) - (c_1 - c_2 + c_3) = -c_4$

So, in general, we have $s_{2n} - s_{2n-1} = c_{2n}$, or

$$s_{2n} = s_{2n-1} + c_{2n}.$$

Since by assumption, $\lim_{n\to\infty} c_n = 0$ regardless of whether $n = 2n$ (even) or $n = 2n - 1$ (odd). Hence, by using the limit laws and the equation above, we have

$$\begin{aligned} \lim_{n\to\infty} s_{2n} &= \lim_{n\to\infty} s_{2n-1} + \lim_{n\to\infty} c_{2n} \\ \Rightarrow \underbrace{\lim_{n\to\infty} s_{2n}}_{L} &= \underbrace{\lim_{n\to\infty} s_{2n-1}}_{L'} + 0 \\ \Rightarrow \qquad L &= L'. \end{aligned}$$

Thus the sequence of partial sums converges to a unique limit and the corresponding alternating series converges to that limit. †

3. **Theorem: Alternating Series Remainder**

Let $R_n = |S - s_n|$ be the remainder in approximating the value of a convergent alternating series $\sum_{n=1}^{\infty}(-1)^n c_n$ by the sum of the first n terms (s_n). Then the absolute value of the remainder R_N involved in approximating the sum S by the sequence of partial sums, s_n, is less than or equal to the first neglected term. ie.

$$|R_n| \leq c_{n+1}.$$

4. **Definition: (Absolute Convergence)** We say that the series $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if $\sum_{n=1}^{\infty} |a_n|$ converges. (ie. the sum is a finite number.)

5. **Theorem: (Absolute Convergence Test)**

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Note: Since $a_n = (-1)^{n+1}c_n$ or $a_n = (-1)^n c_n$, $|a_n| = c_n$. So, for the *Absolute Convergene Test*, all you have to do is to test for convergence or divergence of the series, $\sum_{n=1}^{\infty} c_n$. Another words,

$\star$ If $\sum_{n=1}^{\infty} c_n$ converges, then the alternating series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(-1)^n c_n$ converges.

Proof: Note that for each n,

$$-|a_n| \leq a_n \leq |a_n|, \quad \text{hence, by adding } |a_n| \text{ to each term,} \quad 0 \leq a_n + |a_n| \leq 2|a_n|.$$

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} 2|a_n|$ (twice the $|a_n|$) converges. Hence, by the Direct Comparison Test, the non-negative series, $\sum_{n=1}^{\infty}(|a_n|+a_n)$ converges (apply the sum to the inequality listed above).

Note that a_n can be written as $a_n = (|a_n| + a_n) - |a_n|$ by algebraic manipulations. Applying the sum to this equation, we have

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \Big((|a_n| + a_n) - |a_n|\Big)$$

$$= \underbrace{\sum_{n=1}^{\infty}(|a_n| + a_n)}_{\text{converges by the argument above}} - \underbrace{\sum_{n=1}^{\infty} |a_n|}_{\text{converges by the assumption of the theorem}}$$

Therefore, $\sum_{n=1}^{\infty} a_n$ converges. †

6. **Definition: (Conditional Convergence)** A series that converges but does NOT absolutely converges is called **conditionally convergent.** ie. $\sum_{n=1}^{\infty} a_n$ converges conditionally, if

$\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges.

♣ **Example:** Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ **(Alternating Harmonic Series)**. By applying the Alternating Series Test, we have $c_n = \frac{1}{n}$. Note that

1. c_n is decreasing for all $n \geq 1$, and
2. $\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the Alternating Series Test. To classify its convergence (**Absolute or Conditional**), we compute $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} c_n$.

Note that $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{1}{n}$ $\Rightarrow$ harmonic series $\Rightarrow$ Diverges.

Since $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges, we conclude that

Alternating Harmonic series is conditionally convergent. ♣

⋆⋆ **Summary Note:** A series can either be

1. $\sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges. (**Absolutely Convergent**)
2. $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges. (**Conditionally Convergent**)
3. $\sum_{n=1}^{\infty} a_n$ diverges. (**Divergent**)