Calculus II - Chapter 2.7 - Power Series:

(Series that contain x's. The main reason why we studied/cried over infinite series.)

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We learned how to test for convergence of infinite series in the previous sections. The main focus now is to test for convergence of infinite polynomials. We call these polynomials *power series* because they are defined as infinite series of powers of variable, x. Like the good ole' polynomials, power series (infinite polynomials) can be added, subtracted, multiplied, differentiated, and integrated to give new power series.

1. **Definition (Power Series):** If x is a variable, then an infinite series of the form

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n x^n && (1) \\ &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots && (2) \end{aligned}$$

is called a **power series**. More generally, an infinite series of the form

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n (x-c)^n && (3) \\ &= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n + \cdots && (4) \end{aligned}$$

is called a **power series centered at c**, where c=constant. The first equation (1) is the special case obtained by taking $c = 0$ in equation (3).

NOTE: To simplify the notation for power series, we need to agree that $(x-c)^0 = 1$, even if $x = c$. (ie. $0^0 = 1$)

2. **Definition (Domain of a Series):** Let $f(x)$ be a power series centered at c. Then we way that the **domain of f(x)**,

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n,$$

is set of all x for which the power series converges. The set of all x values for which the power series converge is called the **interval of convergence.**

3. **Theorem (Convergence of a Power Series):**

Let $R > 0$ be a real number. The convergence set for a power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ is always an interval of one of the following three types:

1. A single point $x = c$. In this case, the Radius of Convergence is R=0.

2. An interval $(c-R,\, c+R)$ (ie. $|x-c|<R$), plus possible one or both endpoints. In this case, the Radius of Convergence is R itself.

3. The whole real line. In this case, the Radius of Convergence is $R=\infty$.

Furthermore, a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges absolutely on the interior of its interval of convergence, and outside of the given interval, the power series diverges.

Note: The interval of convergence may be open, closed, or half-open, depending on the particular series. At points x with $|x-c|<R$, the series converges absolutely. If the series converges for all values of x, then we say its radius of convergence is infinite. If it converges only at $x=c$, then we say its radius of convergence is 0.

♠ **Example:** A power series centered at $c=2$ with a radius of convergence $R=5$ converges for all x-values up to 5 units away from $x=2$: $(2-5, 2+5)=(-3,7)$. ♠

♣ **Example:** Taking all the coefficients in the first equation (1) from the definition, we obtain a power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

This is the geometric series (instead of r, we have x instead), and it converges to $\frac{1}{1-x}$ (geometric series converges to $a/(1-r)$ provided that $|r|<1$), provided that $|x|<1 \;\Rightarrow\; -1<x<1$. Therefore, the domain of this power series is $(-1,1)$, which we say that the **radius of convergence** is $R=1$, and the **interval of convergence** is $(-1,1)$. Meaning, if you take any values of x in the interval $(-1,1)$, the series converges.

Also, we conclude that

$$\underbrace{\frac{1}{1-x}}_{\text{not a polynomial}} \quad = \quad \underbrace{1 + x + x^2 + x^3 + \cdots + x^n + \cdots,}_{\text{polynomial}} \quad \text{for } -1<x<1.$$

Another way of looking at it, we think of the partial sums of the series on the right as polynomials, call it $P_n(x)$, that approximate the non-polynomial function on the left. (Note that if $P(x)$ is a polynomial, it is smooth, continuous, and differentiable everywhere. So, $1/(1-x)$ is not a polynomial function because it has an infinite discontinuity at $x=1$.) ♣

4. **How to Test a Power Series for Convergence:** If the series does not follow the geometric series format, to find the radius of convergence and the interval of convergence, we do the following:

1. Use the *Ratio Test (or Root Test)* to find the interval where the series converges absolutely. Ordinarily, this is an open interval

$$|x - c| < R \quad \text{or} \quad c - R < x < c + R$$

Then we call $R =$ *radius of convergence*, and $(c - R, c + R) =$ *interval of convergence.*

2. If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint of the interval of convergence obtained. (We usually use geometric, p-series, comparison, integral, alternating series tests to determine our conclusion.)

3. If the interval of absolute convergence is $(c - R, c + R)$, the series diverges for all values of x that's outside of the interval (it does not even converge conditionally), because the n-th term does not approach 0 for those values of x.

5. **Theorem (Properties of Functions Defined by Power Series):**

Suppose that $f(x)$ is the sum of a power series on on interval $(c - R,\ c + R)$. That is

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n + \cdots$$

Then $f(x)$ is differentiable (hence continuous) on the interval $(c - R, c + R)$ and

1.

$$\begin{aligned} f'(x) &= \sum_{n=0}^{\infty} \frac{d}{dx}[a_n(x-c)^n] \\ &= \sum_{n=1}^{\infty} n a_n (x-c)^{n-1} \\ &= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots \end{aligned}$$

2.

$$\begin{aligned} \int f(x)\,dx &= \int \sum_{n=0}^{\infty} a_n(x-c)^n\,dx \\ &= \sum_{n=0}^{\infty} a_n \int (x-c)^n\,dx \\ &= \sum_{n=0}^{\infty} \frac{a_n(x-c)^{n+1}}{n+1} + C \\ &= C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1} \\ &= C + a_0(x-c) + a_1\frac{(x-c)^2}{2} + a_2\frac{(x-c)^3}{3} + \cdots \end{aligned}$$

The radius of convergence of the $f'(x)$ and $\int f(x)dx$ is the same as that of the original power series. However, **the interval of convergence may be different at the end points**.